

Holographic Fermi Surfaces and Entanglement Entropy

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Abstract

We argue that Landau-Fermi liquids do not have any gravity duals in the purely classical limit. We employ the logarithmic behavior of entanglement entropy to characterize the existence of Fermi surfaces. By imposing the null energy condition, we show that the specific heat always behaves anomalously. We also present a classical gravity dual which has the expected behavior of the entanglement entropy and specific heat for non-Fermi liquids.

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1 Introduction and Summary

The AdS/CFT correspondence [1] provides us a very useful tool to study various properties of strongly coupled systems. It is clear that among many possible states in condensed matter physics, systems with Fermi surfaces are remarkably important. Recently, there have been exciting progresses on the holographic realizations of Fermi surfaces using the AdS/CFT such as the ones using the Reissner-Nordström black holes [2–5] and the star-like solutions, called electron stars [6–8]. Refer also to [9–12] for further developments and other constructions. The former construction leads to the non-Fermi liquids, while the latter to the standard Landau-Fermi liquids. The non-Fermi liquids have anomalous behaviors, for example, with respect to the specific heat and conductivity when we compare them with the Landau-Fermi liquids.

An outstanding advantage of AdS/CFT is that we can analyze strongly coupled systems in a classical gravity by ignoring all quantum corrections. However, there have not been found any holographic constructions of Fermi surfaces where basic quantities such as the free energy, the entropy and the specific heat are all calculated in the classical gravity and agree with the expected standard behaviors of Fermi surfaces. For example, a charged black hole whose near horizon includes AdS_2 has the unusual property of the non-vanishing entropy even at zero temperature. On the other hand, in the electron star solutions, which approach to the Lifshitz geometry [13] in the IR limit, the specific heat and entropy are not large enough to be calculated in the classical gravity. The purpose of this paper is to study if it is possible to construct a holographic dual of Fermi surfaces in a purely classical gravity. If such a gravity dual exists, we expect that the number of Fermi surfaces is of order N^2 , taking the Fermi energy to be order one. Here we are considering a system where fermions belong to the adjoint representation of the $SU(N)$ gauge group. The fractionalized fermi liquid [9] is a promising candidate of such $O(N^2)$ fermi surfaces.

For this purpose, we need to make the definition of Fermi surfaces clear. In this paper, we employ the entanglement entropy [14–17] to characterize the presence of Fermi surface. We define systems with Fermi surfaces by requiring that *their entanglement entropies show the logarithmic violation of the area law*. This has been shown explicitly for free fermion theories [18] and recent progresses in condensed matter physics support that this property is true even for non-Fermi liquids [19, 20].

By using the holographic entanglement entropy [21–23], we can rewrite this requirement as the IR behavior of the metric in the gravity dual. We mainly focus on 2+1 dimensional systems dual to asymptotic AdS_4 geometries. Then we require the null en-

ergy condition, which is expected to be satisfied by any physically sensible backgrounds with well-defined holographic duals. This leads to additional constraints on the metric. Eventually, we find that to satisfy these two requirements the specific heat C should behave like $C \propto T^\alpha$ with the constraint $\alpha \leq 2/3$ at low temperature T for 2+1 dimensional systems. Thus the standard Fermi liquid ($\alpha = 1$) is not allowed. In this way, we find that under our classical gravity dual assumption, we can construct only non-Fermi liquids. We also give an example of an effective gravity action which has such a gravity solution. We will leave the embeddings of our gravity backgrounds in string theory as a future problem.

This paper is organized as follows: In section two, we discuss the behavior of entanglement entropy in the presence of Fermi surfaces. In section three, we study the constraints on the gravity dual metric for the Fermi surfaces from the viewpoint of the entanglement entropy. In section four, we further impose the null energy condition and see how it constrains the behavior of the specific heat. In section five, we present an effective gravity model which has the expected behavior of the entanglement entropy and specific heat for non-Fermi liquids.

2 Fermi Surface and Entanglement Entropy

Consider the entanglement entropy in a $d+1$ dimensional quantum field theory (QFT) on $R^{1,d}$. The coordinate of $R^{1,d}$ is denoted by $(t, x_1, x_2, \dots, x_d)$. The entanglement entropy S_A associated with a subsystem A is defined by the von-Neumann entropy $S_A = -\text{Tr} \rho_A \log \rho_A$. ρ_A is the reduced density matrix defined in terms of the total density matrix ρ by $\rho_A = \text{Tr}_B \rho$, where the subsystem B is the complement of A .

We choose the subsystem A as a strip with the width l . This is explicitly expressed as

$$A = \{(x_1, x_2, \dots, x_d) | -\frac{l}{2} \leq x_1 \leq \frac{l}{2}, \quad 0 \leq x_2, x_3, \dots, x_d \leq L\}. \quad (2.1)$$

In a $d+1$ dimensional conformal field theory, the entanglement entropy S_A behaves like

$$S_A = \gamma \frac{L^{d-1}}{\epsilon^{d-1}} - \alpha \frac{L^{d-1}}{l^{d-1}}, \quad (2.2)$$

where ϵ is the UV cut off ($\epsilon \rightarrow 0$ in the UV limit); γ and α are numerical constants which represent the degrees of freedom [16, 17]. This has been confirmed in the holographic calculation [21, 22]. The leading divergent term is proportional to the area of the boundary of A and this is called the area law [14, 15]. This is true for all QFTs with UV fixed points.

2.1 Logarithmic Violation of Area Law

On the other hand, for a $d + 1$ dimensional QFT with a Fermi surface, we expect that the finite part of S_A is substantially modified when the size l of the subsystem A is large enough

$$S_A = \gamma \frac{L^{d-1}}{\epsilon^{d-1}} + \eta L^{d-1} k_F^{d-1} \log(l k_F) + O(l^0), \quad (2.3)$$

where k_F is the Fermi momentum and η is a positive numerical constant.

Before we give a field theoretic explanation of the behavior (2.3) below, we would like to point out that (2.3) is essentially the same as the logarithmic violation of area law in fermionic theories, which is well known in the condensed matter literature.

In the papers [18] (see also the review [15]), the violation of area law in free fermions on lattices has been shown in any dimension d . They argue that the leading term in the entropy looks like

$$S_A \sim L^{d-1} \log L + \dots, \quad (2.4)$$

where we assume that the subsystem A is given by a finite size (but large) region (e.g. a d dimensional ball) which spans order L in any directions. Their calculation is based on a discretized lattice setup, and there they assume that the Fermi surface has a finite area. In our continuum limit of the quantum field theories, this corresponds to the case where the Fermi energy is the same order of the UV cut off scale i.e. $k_F \sim \epsilon^{-1}$. Indeed, in this limit, the second term in (2.3) gets dominant and leads to the violation of the area law (2.4) by setting $l \sim L$. However, in this paper what we are interested in is the case where k_F is finite while ϵ is taken to be zero. Therefore we have the expression (2.3) instead of (2.4).

Even in the presence of strong interactions this kind of the violation of area law has been expected. The recent calculation of S_A for a critical spin liquid state [20] can be regarded as an evidence of such a log term even in a strongly coupled fermion theory with a Fermi surface. Also such a violation of area law has also been argued in [19].

2.2 Free Fermion Calculation

To understand (2.3), we would like to consider a calculation of S_A for a free fermion in $(d + 1)$ dimension⁴. We compactify x_1, x_2, \dots, x_d on a torus with a large radius L . The momentum in these directions are quantized as

$$k_i = \frac{n^i}{L}, \quad (i = 1, 2, \dots, d). \quad (2.5)$$

⁴Similar estimations have already been done in [19, 23]. See also [24] for an explicit calculation.

We assume there exists a Fermi surface with the Fermi momentum k_F and the ground state is describes as

$$|\Psi\rangle = \prod_{|k| < k_F} b_{n_1, n_2, \dots, n_d}^\dagger |0\rangle. \quad (2.6)$$

Since the theory is free, we can treat sectors with different quantum numbers $\vec{n} \equiv (n_2, n_3, \dots, n_d)$ as decoupled independent sectors. Each sector with a fixed number of \vec{n} can be regarded as a $1 + 1$ dimensional free massive fermion theory with the Fermi surface at $k_1 = \sqrt{k_F^2 - \vec{k}^2}$. The mass is given by $m = |\vec{k}|$. The ground state in the $1 + 1$ dimensional theory for a fixed \vec{n} looks like

$$|\Psi_{\vec{n}}\rangle = \prod_{k_1 \leq \sqrt{k_F^2 - |\vec{k}|^2}} b_{n_1, \vec{n}}^\dagger |0\rangle, \quad (2.7)$$

and the total ground state is given by

$$|\Psi\rangle = \prod_{\vec{n}} |\Psi_{\vec{n}}\rangle. \quad (2.8)$$

Notice that when $k_F < |\vec{k}|$ there is no Fermi surface in the reduced two dimensional fermion theory.

Since the density matrix is factorized into sectors with various \vec{n} , we can express S_A as the summation over each sector:

$$S_A^{(d+1)} = \sum_{\vec{n}} S_A^{(1+1)}(\vec{n}). \quad (2.9)$$

We are interested in the region

$$l \gg \frac{1}{k_F}, \quad (2.10)$$

so that the entanglement entropy can probe the IR physics. When $|\vec{k}| > k_F$, there is no Fermi surface and $S_A^{(1+1)}(\vec{n})$ can be found from the known result in two dimensional CFT. If we define $S_A^{QFT_2}$ to be the entanglement entropy for a massive two dimensional QFT without any Fermi surface, then it behaves like

$$\begin{aligned} S_A^{QFT_2} &\sim \log l/\epsilon \quad (lm \ll 1), \\ S_A^{QFT_2} &\sim -\log m\epsilon \quad (lm \gg 1). \end{aligned} \quad (2.11)$$

Note also that we omitted the coefficient, which is proportional to the central charge [16].

Since $l \gg \frac{1}{k_F}$, when $|\vec{k}| > k_F$ we find

$$S_A^{(1+1)}(\vec{n}) \sim -\log(|\vec{k}|\epsilon). \quad (2.12)$$

In this way we obtain the following estimation

$$\begin{aligned}
S_A^{(d+1)} &\sim L^{d-1} \int_{k_F < |\vec{k}| < \epsilon^{-1}} (dk)^{d-1} S_A^F(|\vec{k}|, 0) + L^{d-1} \int_{|\vec{k}| < k_F} (dk)^{d-1} S_A^F(|\vec{k}|, \sqrt{k_F^2 - |\vec{k}|^2}), \\
&\sim -L^{d-1} \int_{k_F < |\vec{k}| < \epsilon^{-1}} (dk)^{d-1} \log(\epsilon |\vec{k}|) + L^{d-1} \int_{|\vec{k}| < k_F} (dk)^{d-1} S_A^F(|\vec{k}|, \sqrt{k_F^2 - |\vec{k}|^2}), \quad (2.13)
\end{aligned}$$

where $S_A^F(m, \mu)$ is the entanglement entropy for a free massive fermion in $1+1$ dimension with the mass m and the Fermi momentum μ . The first term in (2.13) comes from the modes $|\vec{k}| > k_F$ can be simplify estimated as

$$-L^{d-1} \int_{k_F < |\vec{k}| < \epsilon^{-1}} (dk)^{d-1} \log(\epsilon |\vec{k}|) \sim \left(\frac{L}{\epsilon}\right)^{d-1} + \dots, \quad (2.14)$$

which just represents the standard area law.

To proceed further we need to know S_A^F for non-zero values of μ . The dispersion relation in the massive two dimensional fermion obtained by the dimensional reduction looks like

$$\begin{aligned}
E_k &= \sqrt{|\vec{k}|^2 + \left(\sqrt{k_F^2 - |\vec{k}|^2} + \delta k_1\right)^2}, \\
&\simeq k_F + \frac{\sqrt{k_F^2 - |\vec{k}|^2}}{k_F} \delta k_1. \quad (2.15)
\end{aligned}$$

Therefore near the Fermi surface (i.e. $\delta k_1 \ll k_F$), the fermion behaves like a massless fermion. Therefore we can estimate the remained part of S_A as follows

$$\begin{aligned}
&L^{d-1} \int_{|\vec{k}| < k_F} (dk)^{d-1} S_A^F(|\vec{k}|, \sqrt{k_F^2 - |\vec{k}|^2}) \\
&\sim L^{d-1} \int_{|\vec{k}| < k_F} (dk)^{d-1} \log l/\epsilon. \quad (2.16)
\end{aligned}$$

This is because the low energy excitation around the Fermi surface becomes relativistic as in (2.15).

Therefore, by combining (2.14) and (2.16), we find that the total behavior of S_A looks like

$$S_A^{(d+1)} \sim \left(\frac{L}{\epsilon}\right)^{d-1} + L^{d-1} k_F^{d-1} \log(l k_F) + \dots \quad (2.17)$$

Notice that the logarithmic divergent terms $O(\log \epsilon)$ are obviously canceled between (2.14) and (2.16). In this way, we managed to derive the formula (2.3).

So far we assumed the free fermions. In order to compare the results with those in AdS/CFT, we need to see how this argument changes due to the strong interactions. However, there are evidences which argue that the logarithmic dependence (2.3) does not change. One of them is the recent calculations done for the interacting Fermi surfaces in spin liquids [20]. Another is the fact that though in non-Fermi liquids we often encounter non-relativistic dynamical exponents for the dispersion relation around the Fermi surface, the logarithmic behavior (2.11) does not change (only its coefficient changes). Therefore, in this paper, we define systems with Fermi surfaces by requiring that their entanglement entropies show the logarithmic violation of the area law, as already mentioned.

2.3 Entanglement Entropy for 2D Free Massless Dirac Fermions

It is useful to look at an analytical expression of the entanglement entropy in two dimensional CFT at finite chemical potential. Below we will calculate S_A for a two dimensional free massless Dirac fermion on a circle at finite temperature $T = 1/\beta$ and at finite chemical potential μ . We would like to ask the readers to refer to [25] for details where a similar analysis has been done when $\mu = 0$.

We consider a torus whose lengths in the space and the Euclidean time direction are given by β and 1, respectively. The size of subsystem A is denoted by L such that $0 < L < 1$. The thermal partition function is expressed by using the theta functions

$$Z_{th} = \text{Tr} e^{-\beta H - 2\pi\beta\mu N} = \frac{|\theta_3(i\beta\mu|i\beta)|^2}{|\eta(i\beta)|^2}, \quad (2.18)$$

where H and N are Hamiltonian and the fermion number.

The two point function of twisted operators σ_k in the Z_N orbifolded free fermion theory on a torus is given by

$$\langle \sigma_k(L) \sigma_{-k}(0) \rangle = \left| \frac{2\pi\eta(i\beta)^3}{\theta_1(L|i\beta)} \right|^{4\Delta_k} \cdot \left| \frac{\theta_3(\frac{kL}{N} + i\beta\mu|i\beta)}{\theta_3(i\beta\mu|i\beta)} \right|^2, \quad (2.19)$$

where $\Delta_k = \frac{k^2}{2N^2}$ is the conformal dimension of the k -th twisted operator σ_k . By applying the replica trick, we obtain

$$S_A = -\frac{\partial}{\partial N} [\log \text{Tr}[(\rho_A)^n]] \Big|_{N=1} = -\frac{\partial}{\partial N} \left[\log \prod_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \langle \sigma_k(L) \sigma_{-k}(0) \rangle \right] \Big|_{N=1}. \quad (2.20)$$

In the end, we find the high temperature expansion

$$\begin{aligned}
S_A(L, \beta, \mu) &= \frac{1}{3} \log \left[\frac{\beta}{\pi \epsilon} \sinh \left(\frac{\pi L}{\beta} \right) \right] + \frac{1}{3} \sum_{m=1}^{\infty} \log \left[\frac{(1 - e^{2\pi \frac{L}{\beta}} e^{-2\pi \frac{m}{\beta}})(1 - e^{-2\pi \frac{L}{\beta}} e^{-2\pi \frac{m}{\beta}})}{(1 - e^{-2\pi \frac{m}{\beta}})^2} \right] \\
&+ 2 \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \left(\frac{\frac{\pi l L}{\beta} \coth \left(\frac{\pi l L}{\beta} \right) - 1}{\sinh \left(\frac{\pi l}{\beta} \right)} \right) \cos(2\pi \mu l), \tag{2.21}
\end{aligned}$$

where $\epsilon (\rightarrow 0)$ is the UV cut off.

We find basic properties as follows

$$S_A(L, \beta, \mu) = S_A(L, \beta, 1 - \mu) = S_A(L, \beta, \mu + 1), \tag{2.22}$$

which is explained by the particle-hole exchange symmetry and the periodicity of energy levels peculiar to the two dimensional free fermion theory.

We can also confirm that the difference

$$S_A(L = 1 - \delta, \beta, \mu) - S_A(L = \delta, \beta, \mu) = \frac{\pi}{3\beta} + 2 \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \left(\frac{\frac{\pi l L}{\beta} \coth \left(\frac{\pi l}{\beta} \right) - 1}{\sinh \left(\frac{\pi l}{\beta} \right)} \right) \cos(2\pi \mu l), \tag{2.23}$$

in the limit $\delta \rightarrow 0$ actually coincides with the thermal entropy

$$S_{th}(\beta, \mu) = -\beta^2 \frac{\partial}{\partial \beta} (\beta^{-1} \log Z_{th}). \tag{2.24}$$

In the low temperature expansion entanglement entropy is given by (for $0 \leq \mu < 1$)

$$\begin{aligned}
S_A(L, \beta, \mu) &= \frac{1}{3} \log \left[\frac{1}{\pi \epsilon} \sin(\pi L) \right] + \frac{1}{3} \sum_{m=1}^{\infty} \log \left[\frac{(1 + e^{2\pi i L} e^{-2\pi \beta(m-1/2)})^2 (1 + e^{-2\pi i L} e^{-2\pi \beta(m-1/2)})^2}{(1 + e^{-2\pi \beta(m-1/2)})^2} \right] \\
&+ 2 \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \left(\frac{\pi l L \cot(\pi l L) - 1}{\sinh(\pi \beta l)} \right) \cosh(2\pi \mu \beta l) \tag{2.25}
\end{aligned}$$

Thus at zero temperature, we simply find

$$S_A(L, \infty, \mu) = \frac{1}{3} \log \left[\frac{1}{\pi \epsilon} \sin(\pi L) \right], \tag{2.26}$$

we confirm that the logarithmic behavior still exists even if we turn on the chemical potential as we assumed in the previous calculations.

3 Possible Holographic Duals with Fermi Surfaces

We are interested in holographic duals of strongly coupled $2+1$ dimensional systems with Fermi surfaces. Thus we consider $3+1$ dimensional gravity backgrounds. We consider the following metric

$$ds^2 = \frac{R^2}{z^2} \left(-f(z)dt^2 + g(z)dz^2 + dx^2 + dy^2 \right). \quad (3.27)$$

This is the most general metric when we require the translational and rotational symmetry in x and y directions.⁵ Notice that the metric component g_{tz} can be eliminated by a coordinate transformation. We require that its dual theory has a UV fixed point and therefore (3.27) should be asymptotically AdS_4 . Since $z = 0$ corresponds to the boundary, we require

$$f(0) = g(0) = 1. \quad (3.28)$$

3.1 Holographic Entanglement Entropy

Now we holographically calculate the entanglement entropy in these gravity backgrounds. The holographic entanglement entropy [21–23] is given by

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}, \quad (3.29)$$

where γ_A is the codimension two minimal area surface which coincides with ∂A at the boundary $z = 0$.

We are interested in the strip shape subsystem A (2.1) i.e.

$$A = \{(x, y) | -\frac{l}{2} \leq x \leq \frac{l}{2}, \quad 0 \leq y \leq L\}. \quad (3.30)$$

We can specify the minimal area surface γ_A by the surface $x = x(z)$ in (3.27).

The area of this surface can be found as

$$\text{Area}(\gamma_A) = 2R^2 L \int_{\epsilon}^{z_*} \frac{dz}{z^2} \sqrt{g(z) + x'(z)^2}, \quad (3.31)$$

where z_* is the turning point, where x' gets divergent; ϵ is the UV cut off. The variational principle for $x(z)$ leads to

$$\frac{x'(z)}{z^2 \sqrt{g(z) + x'^2}} = \text{const.} \quad , \quad (3.32)$$

⁵Here we did not assume any breaking of symmetry as we are interested in realizing a fermi liquid in the absence of any external fields. However, it is also possible that the logarithmic behavior of entanglement entropy occurs in the presence of external fields. A good example will be the magnetic quantum critical point found in an asymptotically AdS_5 geometry [26], as pointed out in [27]

which can be solved as

$$x'(z) = \frac{z^2}{z_*^2} \sqrt{\frac{g(z)}{1 - \frac{z^4}{z_*^4}}} . \quad (3.33)$$

The width l is related to z_* by

$$l = 2 \int_0^{z_*} dz \frac{z^2}{z_*^2} \sqrt{\frac{g(z)}{1 - \frac{z^4}{z_*^4}}} . \quad (3.34)$$

In the end, the area (3.31) is expressed as follows

$$\text{Area}(\gamma_A) = 2R^2L \int_\epsilon^{z_*} \frac{dz}{z^2} \sqrt{\frac{g(z)}{1 - \frac{z^4}{z_*^4}}} . \quad (3.35)$$

To obtain explicit results, let us assume that the function $g(z)$ scales as

$$\begin{aligned} g(z) &\simeq \left(\frac{z}{z_F} \right)^{2n} & (z \gg z_F), \\ &\simeq 1 & (z \ll z_F), \end{aligned} \quad (3.36)$$

with a certain scale z_F . The parameter z_F is dual to the length scale where the geometry starts modified and shows peculiar IR behaviors. We assume $n > 1$ below. We are interested in the very IR limit $z_* \gg z_F$ and would like to see the l dependence of the finite part of S_A .

Then (3.34) is estimated as

$$l \sim \frac{2}{z_*^2 z_F^n} \int^{z_*} dz \frac{z^{2+n}}{\sqrt{1 - z^4/z_*^4}} \sim c_n \frac{z_*^{n+1}}{z_F^n}, \quad (3.37)$$

where c_n is a positive numerical constant.

On the other hand, the area is evaluated as

$$\begin{aligned} \text{Area}(\gamma_A) &\simeq \frac{2R^2L}{\epsilon} + \frac{2R^2L}{z_F^n} \int^{z_*} dz \frac{z^{n-2}}{\sqrt{1 - z^4/z_*^4}} \\ &\simeq \frac{2R^2L}{\epsilon} + d_n \frac{R^2L z_*^{n-1}}{z_F^n}, \end{aligned} \quad (3.38)$$

where d_n is a positive numerical constant.

In the end, the behavior of S_A is obtained as follows (k_n is a positive constant)

$$S_A = \frac{R^2L}{2G_N^{(4)}\epsilon} + k_n \frac{R^2}{G_N} \frac{L}{z_F} \cdot \left(\frac{l}{z_F} \right)^{\frac{n-1}{n+1}} + \dots, \quad (3.39)$$

where the omitted term represents the subleading term in the limit $z_* \gg z_F$ or equally $l \gg z_F$. The leading divergent term agrees with the area law and this is expected because our background is asymptotically AdS. Note that the maximum increasing rate of S_A as a function of l is linear, which corresponds to $n \rightarrow \infty$ limit. This is consistent with the fact that the maximal allowed entropy is always given by the log of the dimensional of Hilbert space of A , which is clearly proportional to l . This limit is realized in the charge AdS black holes whose the near horizon geometry $\text{AdS}_2 \times \mathbb{R}^2$. In this case, the spacetime has a horizon with non-vanishing entropy and the extensive behavior of S_A is indeed expected.

On the other hand, for the purpose of a gravity dual of a Fermi surfaces, we need to set $n = 1$. In this case, we indeed find the following behavior (notice that (3.37) remains the same, while we need to modify (3.38))

$$S_A = \frac{R^2 L}{2G_N^{(4)} \epsilon} + k_1 \frac{R^2}{G_N} \frac{L}{z_F} \log \left(\frac{l}{z_F} \right) + O(l^0), \quad (3.40)$$

which agrees with (2.3). Therefore the existence of Fermi surfaces requires the behavior

$$\begin{aligned} g(z) &\simeq \left(\frac{z}{z_F} \right)^2 & (z \gg z_F), \\ &\simeq 1 & (z \ll z_F). \end{aligned} \quad (3.41)$$

The parameter z_F is now interpreted as the scale of the Fermi surface $\sim k_F^{-1}$. More precisely, since we are expecting the existence of many fermi surfaces, z_F is regarded as the average of their fermi levels. This is our basic guiding principle to search a holographic dual of Fermi surfaces. It is also straightforward to extend the above argument to higher dimensions.

3.2 Circular Subsystems

So far we analytically confirmed the logarithmic dependence on the subsystem size l only for the strip subsystems. Here we would like to study the case where the subsystem A is given by a circular disk with radius l . In conformal field theories, we obtain

$$S_A = \gamma \cdot \frac{l}{\epsilon} - \delta, \quad (3.42)$$

where γ and δ are constants [17, 21, 28]. In the presence of the Fermi surfaces, we again expect

$$S_A = \gamma \cdot \frac{l}{\epsilon} + \tilde{\eta} \cdot l k_F \log l k_F + \dots \quad (3.43)$$

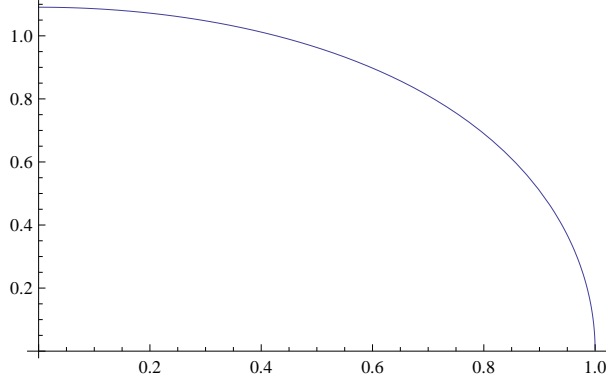


Figure 1: The shape of the function $r(z)$ for $z_* = 1$.

In the holographic calculations with the metric (3.27), we introduce a polar coordinate such that $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$. The subsystem A of the CFT corresponds to $r \leq l$ at $z = 0$ in the gravity background. The surface γ_A is specified by $r = r(z)$ in (3.27) at a fixed time t . The holographic entanglement entropy is given by minimizing

$$S_A = \frac{\pi R^2}{2G_N} \int_{\epsilon}^{z_*} \frac{dz}{z^2} r(z) \sqrt{g(z) + r'(z)^2}. \quad (3.44)$$

The boundary condition of $r(z)$ is $r(0) = l$ and $r(z_*) = 0$. At the turning point $z = z_*$, $r'(z)$ gets divergent. The equation of motion for $r(z)$ is given by

$$\partial_z \left(\frac{r(z)r'(z)}{z^2 \sqrt{g(z) + r'(z)^2}} \right) = \frac{\sqrt{g(z) + r'(z)^2}}{z^2}. \quad (3.45)$$

The behavior near $z = z_*$ is completely fixed by the equation of motion (3.45) and given in the form

$$r(z) \simeq (z_* - z)^{1/2} (r_0 + r_1(z_* - z) + \dots). \quad (3.46)$$

We are working for the choice

$$g(z) = \sqrt{1 + z^4/z_F^4}, \quad (3.47)$$

which is asymptotically AdS_4 and leads to the IR geometry (3.41). Since we need to perform the numerical analysis below, we will simply set $\frac{\pi R^2}{2G_N} = 1$.

The explicit form of $r(z)$ is shown in Fig.1. We plotted the finite part of the entanglement entropy in Fig.2 for the choice $z_F = 1$. Its behavior is indeed well approximated by

$$S_A^{fin} \simeq 0.50 \cdot l \log l - 0.60 \cdot l. \quad (3.48)$$

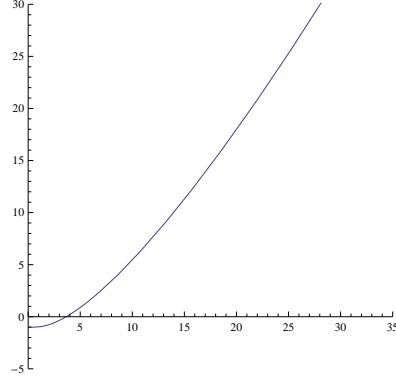


Figure 2: The finite part of S_A (called S_A^{fin}) as a function l .

By looking at the behavior of the coefficients in (3.48), we can numerically confirm the dependence of $z_F \propto k_F^{-1}$ as in (3.43).

4 Null Energy Condition and Specific Heat

To have a physically sensible gravity theory, we need to require an appropriate energy condition. In the presence of negative cosmological constants, we usually impose the null energy condition:

$$T_{\mu\nu}N^\mu N^\nu \geq 0, \quad (4.49)$$

where N^μ denotes any null vector. For example, we employ this condition to prove the holographic c-theorem [29].

We find from the Einstein equation that the energy stress tensor $T_{\mu\nu}$ is simply given by

$$T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \quad (4.50)$$

We can choose N^μ as

$$N^t = \frac{1}{\sqrt{f(z)}}, \quad N^z = \frac{\cos \theta}{\sqrt{g(z)}}, \quad N^x = \sin \theta, \quad (4.51)$$

where θ is an arbitrary constant. Then

$$\begin{aligned} T_{\mu\nu}N^\mu N^\nu &= -\frac{zg(z)f'(z)^2 + f(z)(zf'(z)g'(z) + g(z)(4f'(z) - 2zf''(z)))}{4zf(z)^2g(z)^2} \sin^2(\theta) \\ &\quad - \frac{g(z)f'(z) + f(z)g'(z)}{zf(z)g(z)^2} \cos^2(\theta), \end{aligned} \quad (4.52)$$

and so the null energy condition (4.49) is satisfied if and only if

$$\begin{aligned} g(z)f'(z) + f(z)g'(z) &\leq 0, \\ zg(z)f'(z)^2 + f(z)(zf'(z)g'(z) + g(z)(4f'(z) - 2zf''(z))) &\leq 0. \end{aligned} \quad (4.53)$$

Let us focus on the IR geometry. If we assume the behavior

$$f(z) \propto z^{-2m}, \quad g(z) \propto z^{2n}, \quad (4.54)$$

then the conditions (4.53) lead to

$$m \geq n. \quad (4.55)$$

4.1 Constraints on Specific Heat

Since the Landau-Fermi liquid has the peculiar property that the specific heat is linearly proportional to the temperature, it is helpful to investigate the specific heat. At finite temperature, the IR geometry (4.54) is modified into a black hole solution of the form:

$$ds^2 = R^2 \left(-z^{-2(m+1)} h(z) dt^2 + z^{2(n-1)} \tilde{h}(z)^{-1} dz^2 + \frac{dx^2 + dy^2}{z^2} \right), \quad (4.56)$$

where we set $R = 1$. We define the location of the horizon to be $z = z_H$. For $\frac{z}{z_H} \ll 1$, the metric should approach (4.54) and therefore we have $h(z) \simeq 1$ and $\tilde{h}(z) \simeq 1$. On the other hand, in the near horizon region $z \simeq z_H$, we generically have the regular non-extremal horizon and thus they behave like

$$h(z) \sim \tilde{h}(z) \sim \frac{z_H - z}{z_H}, \quad (4.57)$$

up to constant factors.

Thus we can estimate the temperature as

$$T \propto z_H^{-m-n-1}, \quad (4.58)$$

by requiring that the Euclidean geometry at $z = z_H$ is smooth. Since the thermal entropy S is given by the area of horizon we find

$$S \propto \frac{V_2}{z_H^2} \propto V_2 \cdot T^{\frac{2}{m+n+1}}, \quad (4.59)$$

where V_2 is the volume of the dual CFT_3 .

The specific heat is found as

$$C = \frac{\partial E}{\partial T} = T \frac{\partial S}{\partial T} \propto T^{\frac{2}{m+n+1}}. \quad (4.60)$$

The existence of Fermi surfaces requires $n = 1$ as explained in (3.41). In this case, the null energy condition (4.55) leads to⁶

$$\frac{2}{m+n+1} \leq \frac{2}{3}. \quad (4.61)$$

Since the standard Fermi liquid has the linear specific heat $C \propto T$, classical gravity duals cannot have the Landau-Fermi liquids. Instead, our specific heat (4.60) suggests that they correspond to non-Fermi liquids⁷.

If the Fermi surface is governed by a two dimensional theory with the dispersion relation $E \sim k^{\hat{z}}$, then we expect

$$C \propto S \propto T^{\frac{1}{\hat{z}}}, \quad (4.62)$$

which agrees with the analysis in [32]. The null energy condition requires

$$\hat{z} \geq \frac{3}{2}. \quad (4.63)$$

In summary, we found that the Landau-Fermi liquids cannot holographically be realized in any purely classical gravity. However, non-Fermi liquids are not excluded as long as the bound (4.63) is satisfied (in $2+1$ dimension). For such a classical gravity dual, we expect the presence of order N^2 Fermi surfaces, taking the Fermi energy to be order one.

It might be useful to note that the electron stars [6–8] are not included in our definition of Fermi surfaces with classical gravity duals. Actually, in an electron star, since there is a smaller number ($\ll N^2$) of Fermi surfaces [7], the entanglement entropy and specific heat contributed by them only appear as quantum corrections to the classical gravity calculations (see [8] for the analysis of the specific heat). On the other hand, the free energy is computable in classical gravity because the chemical potential is taken to be very large as $O(\sqrt{N})$ in the electron star. The same is also true for the IR gapped models [12].

5 Effective Gravity Model

Finally, we would like to give an effective gravity model which has the logarithmic behavior (2.3) of the holographic entanglement entropy, peculiar to systems with Fermi surfaces.

⁶In AdS_5 gravity duals, we find $C \propto T^{\frac{3}{m+n+1}}$ and the null energy condition requires $m \geq n$. The Fermi surface (or equally the logarithmic entanglement entropy) corresponds to $n = 2$. Therefore we find the behavior $C \propto T^\alpha$ with the constraint $\alpha \leq \frac{3}{5}$ for the $3+1$ dimensional fermi surfaces.

⁷It might be intriguing to note that a $2+1$ dimensional Kondo breakdown quantum critical point (a non Fermi liquid) has $C \sim T^{2/3}$ behavior [30]. Also the same behavior also appear in a class of non-Fermi liquids in the large N limit [31].

We start with the effective action of Einstein-Maxwell-scalar theory:

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[(R - 2\Lambda) - W(\phi) F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \quad (5.64)$$

where $W(\phi)$ and $V(\phi)$ can be any function. $\Lambda = -\frac{3}{R^2}$ is the negative cosmological constant. We would like to mention that this system has been extensively studied in [33–35] and some of our solutions, especially our scaling solutions discussed in section 5.1, correspond to their special examples.

5.1 Equation of Motions and General Solutions

The equations of motion are summarized as follows. The Einstein equation reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{W(\phi)}{2} F^{\rho\sigma} F_{\rho\sigma} g_{\mu\nu} + 2W(\phi) F_\mu^\rho F_{\nu\rho} - \frac{1}{4} \partial_\rho \phi \partial^\rho \phi g_{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} V(\phi) g_{\mu\nu}. \quad (5.65)$$

The Maxwell equation is given by

$$\partial_\mu (\sqrt{-g} W(\phi) F^{\mu\nu}) = 0. \quad (5.66)$$

Finally, the scalar equation of motion reads

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi) - \frac{\partial V(\phi)}{\partial \phi} - \frac{\partial W(\phi)}{\partial \phi} F^{\mu\nu} F_{\mu\nu} = 0. \quad (5.67)$$

Our ansatz for solutions is given as follows: The metric is again given by (3.27). The dilaton and the gauge potential are functions of z

$$\phi = \phi(z), \quad A_t = a(z). \quad (5.68)$$

The Maxwell equation (5.66) leads to

$$a'(z) = \frac{A}{W(\phi)} \sqrt{f(z)g(z)}, \quad (5.69)$$

where A is an integration constant. Notice that A is interpreted as the charge density in the dual quantum field theory.

The Einstein equations (5.65) can be written as

$$\begin{aligned} G_{tt} &= \frac{R^2}{z^2} f(z) \left(\Lambda + \frac{V(\phi)}{2} \right) + \frac{A^2 z^2}{R^2 W(\phi)} f(z) + \frac{f(z)}{4g(z)} \phi'(z)^2, \\ G_{zz} &= -\frac{R^2}{z^2} g(z) \left(\Lambda + \frac{V(\phi)}{2} \right) - \frac{A^2 z^2}{R^2 W(\phi)} g(z) + \frac{1}{4} \phi'(z)^2, \\ G_{xx} &= -\frac{R^2}{z^2} \left(\Lambda + \frac{V(\phi)}{2} \right) + \frac{A^2 z^2}{R^2 W(\phi)} - \frac{1}{4g(z)} \phi'(z)^2. \end{aligned} \quad (5.70)$$

The scalar field equation of motion is given by

$$\frac{z^4}{R^4 \sqrt{f(z)g(z)}} \partial_z \left(\frac{R^2}{z^2} \sqrt{\frac{f(z)}{g(z)}} \partial_z \phi(z) \right) - \frac{\partial V(\phi)}{\partial \phi} + \frac{2A^2 z^4}{R^2 W(\phi)^2} \frac{\partial W(\phi)}{\partial \phi} = 0. \quad (5.71)$$

Now, for our background $G_{\mu\nu}$ are found to be

$$\begin{aligned} G_{tt} &= -\frac{f(z)(3g(z) + zg'(z))}{z^2 g(z)^2}, \\ G_{zz} &= \frac{3f(z) - zf'(z)}{z^2 f(z)}, \\ G_{xx} &= -\frac{z^2 g(z) f'(z)^2 - 4f(z)^2 (3g(z) + zg'(z)) + zf(z)(zf'(z)g'(z) + g(z)(4f'(z) - 2zf''(z)))}{4z^2 f(z)^2 g(z)^2}, \end{aligned} \quad (5.72)$$

From (5.72) and (5.70), the three Einstein equations are equivalently expressed as follows

$$\begin{aligned} V(\phi) &= \frac{z^2 g(z) f'(z)^2 + 4f(z)^2 (-6g(z) + 6g(z)^2 - zg'(z)) + zf(z)(zf'(z)g'(z) + g(z)(4f'(z) - 2zf''(z)))}{4R^2 f(z)^2 g(z)^2}, \\ \frac{1}{W(\phi)} &= -\frac{R^2}{8A^2} \cdot \frac{zg(z)f'(z)^2 + f(z)(zf'(z)g'(z) + g(z)(4f'(z) - 2zf''(z)))}{z^3 f(z)^2 g(z)^2}, \\ \phi'(z)^2 &= -\frac{2(g(z)f'(z) + f(z)g'(z))}{zf(z)g(z)}. \end{aligned} \quad (5.73)$$

It is immediate to find the physical conditions by requiring $W(\phi) \geq 0$ and $\phi'(z)^2 \geq 0$, and we find that they are exactly equivalent to the null energy conditions (4.53)

We can show that (5.71) is automatically satisfied if we substitute (5.73) into it. Thus, in principle, we can always have a full solution to all equations of motion by choosing $V(\phi)$ and $W(\phi)$ appropriately. Only non-trivial constraint comes from the no-ghost (and null energy) condition (4.53).

5.2 IR Solution with Logarithmic Entanglement Entropy

To reproduce (2.3) we need to have

$$g(z) = \frac{z^2}{z_F^2}, \quad (5.74)$$

in the IR region $z \gg z_F$. We also assume the following form of $f(z)$:

$$f(z) = kz^{-p}, \quad (5.75)$$

where k is a positive constant. In this case, the solution exists for $p > 2$ and it is given by

$$\phi(z) = \sqrt{2(p-2)} \log z, \quad (5.76)$$

$$V(\phi) = \frac{6}{R^2} - \frac{(32 + 12p + p^2)z_F^2}{4R^2 z^2}, \quad (5.77)$$

$$W(\phi) = \frac{8A^2 z^6}{z_F^2 p(8+p)R^2}. \quad (5.78)$$

Therefore we find

$$\begin{aligned} V(\phi) &= \frac{6}{R^2} - \frac{(32 + 12p + p^2)z_F^2}{4R^2} e^{-\sqrt{\frac{2}{p-2}}\phi}, \\ W(\phi) &= \frac{8A^2}{z_F^2 p(p+8)R^2} e^{\frac{6}{\sqrt{2(p-2)}}\phi}. \end{aligned} \quad (5.79)$$

It is clear that this solution gets singular in the IR limit $z = \infty$. In order to show that the singularity at zero temperature is not problematic, we need to find regular black hole solutions at finite temperature. We assume the potentials $V(\phi)$ and $W(\phi)$ are given by (5.77) and (5.78). Then we can find the following simple black hole solutions

$$\begin{aligned} g(z) &= \frac{z^2}{z_F^2 h(z)}, \\ f(z) &= k \cdot z^{-p} \cdot h(z), \\ \phi(z) &= \sqrt{2(p-2)} \log z, \end{aligned} \quad (5.80)$$

where $h(z)$ is given by

$$h(z) = 1 - M z^{\frac{p+8}{2}}. \quad (5.81)$$

The positive parameter M represents the mass of the black hole and the horizon is situated at $h(r) = 0$. In this example, we can show that the specific heat C or equally the thermal entropy S behaves as follows

$$C \propto S \propto \left(\frac{T}{z_F \sqrt{k}} \right)^{\frac{4}{p+4}}. \quad (5.82)$$

This temperature dependence is consistent with the general argument in (4.60).

5.3 Embedding into Asymptotically AdS Solutions

Finally, we would like to embed the above IR solution into an asymptotically AdS background. To make the presentation simple, we concentrate on the $p = 3$ case below. Then we can choose

$$g(z) = \sqrt{1 + \frac{z^4}{z_F^4}}, \quad f(z) = \frac{k \cdot z^{-3}}{1 + k \cdot z^{-3}}. \quad (5.83)$$

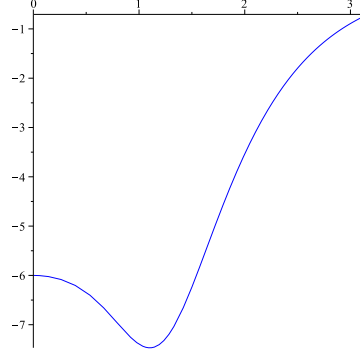


Figure 3: The profile of scalar potential $V(\phi) + 2\Lambda$ for $z_F = k = 1$ and $p = 3$. The unit of the vertical axis is $1/R^2$. The initial condition of the scalar field is set to be $\phi(0) = 0$.

The scalar field is found to be

$$\phi(z) = \sqrt{2} \int_0^z dz \sqrt{\frac{z(3z_F^4 - 2kz + z^4)}{(k + z^3)(z_F^4 + z^4)}}. \quad (5.84)$$

In order to keep the inside of the square root positive for all z , we need to require

$$k < 2z_F^3. \quad (5.85)$$

The scalar field behaves like

$$\begin{aligned} \phi(z) &\simeq \frac{2\sqrt{2}}{\sqrt{3k}} z^{3/2} \quad (z \rightarrow 0), \\ \phi(z) &\simeq \sqrt{2} \log z + \phi_0, \end{aligned} \quad (5.86)$$

where ϕ_0 is a certain constant defined by the above asymptotic behavior.

The resulting scalar potential $V(\phi)$ is plotted in Fig.3. It behaves like

$$V(\phi) \simeq -\frac{9}{8R^2} \phi^2, \quad (\phi \rightarrow 0) \quad (5.87)$$

$$V(\phi) \simeq \frac{6}{R^2} - \frac{(32 + 12p + p^2)z_F^2}{4R^2} e^{-\sqrt{2}(\phi - \phi_0)} \quad (\phi \rightarrow \infty). \quad (5.88)$$

Notice that the tachyonic mass (5.87) marginally satisfies the BF bound.

The behavior of W is plotted in Fig.4. It looks like

$$\begin{aligned} W(\phi) &\simeq 2^5 3^{-11/3} \frac{A^2 k^{4/3}}{R^2} \phi^{-4/3}, \quad (\phi \rightarrow 0) \\ W(\phi) &\simeq \frac{8A^2}{33z_F^2 R^2} e^{3\sqrt{2}(\phi - \phi_0)} \quad (\phi \rightarrow \infty). \end{aligned} \quad (5.89)$$

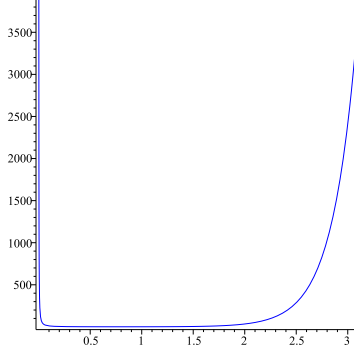


Figure 4: The profile of $W(\phi)$ for $z_F = k = 1$ and $p = 3$. The unit of the vertical axis is A^2/R^2 . The initial condition of the scalar field is set to be $\phi(0) = 0$.

As a function z , W is given by

$$W(z) = \frac{8A^2(k + z^3)^2(z^4 + z_F^4)\sqrt{1 + z^4/z_F^4}}{3R^2z^2(9z_F^4 + 2kz + 11z^4)}. \quad (5.90)$$

It will be an very intriguing future problem to find a string theory embedding which has these properties of V and W .

In summary, in order to have expected solutions, the potentials should behave as follows in the limit $\phi \rightarrow \infty$

$$\begin{aligned} V(\phi) &\simeq \frac{6}{R^2} - \frac{c_1}{R^2}e^{-\sqrt{2}\phi}, \\ W(\phi) &\simeq c_2R^2e^{3\sqrt{2}\phi}, \end{aligned} \quad (5.91)$$

where the dimensionless constants c_1 and c_2 are fixed once we decide a microscopic theory. If we set $k \sim O(z_F^3)$ motivated by (5.85), then we find $\phi_0 \sim -\sqrt{2}\log z_F$. Thus we find the relation (up to a numerical constant)

$$z_F \sim \frac{R}{\sqrt{A}}. \quad (5.92)$$

The gauge potential at the AdS boundary $z = 0$ is found to be (again up to numerical constants)

$$A_t(0) = \int_0^\infty dz \frac{A}{W(z)} \sqrt{f(z)g(z)} \sim \frac{R^2}{Az_F^3} \sim \frac{1}{z_F}. \quad (5.93)$$

This relates the Fermi energy $\mu_F \equiv A_t(0)$ to the charge density A by $\mu_F \propto \sqrt{A}$. In this case, the specific heat (5.82) behaves like

$$C \propto T^{\frac{4}{7}}(\mu_F)^{\frac{10}{7}}. \quad (5.94)$$

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